

## On a square-ice analogue of plane partitions

Suresh Govindarajan<sup>†</sup>, Anthony J. Guttmann<sup>\*</sup> and Varsha Subramanyan<sup>†</sup>

<sup>†</sup>Department of Physics,  
Indian Institute of Technology Madras,  
Chennai 600036, India  
Email: suresh@physics.iitm.ac.in

and

<sup>\*</sup>ARC Centre of Excellence for Mathematics and Statistics of Complex Systems,  
School of Mathematics and Statistics,  
The University of Melbourne,  
Victoria 3010, Australia  
Email: tonyg@ms.unimelb.edu.au

### Abstract

We study a one-parameter family ( $\ell = 1, 2, 3, \dots$ ) of configurations that are square-ice analogues of plane partitions. Using an algorithm due to Bratley and McKay, we carry out exact enumerations in order to study their asymptotic behaviour and establish, via Monte Carlo simulations as well as explicit bounds, that the asymptotic behaviour is similar to that of plane partitions. We finally carry out a series analysis and provide independent estimates for the asymptotic behaviour.

# 1 Introduction

A seller of oranges arranges his oranges in the following fashion. The top layer has a row of  $\ell$  ( $= 1, 2, 3, \dots$ ) oranges, the second layer has oranges forming a  $2 \times (\ell + 1)$  rectangle and in the  $k$ -th layer, the oranges form a  $k \times (k + \ell - 1)$  rectangle (see Figure 1). We call the parameter  $\ell$  the *width* of a configuration. Assuming that there are infinitely many layers, in how many ways can one remove  $n$  oranges without upsetting any other oranges? Denote this by number by  $a_\ell(n)$ . We study properties of the sequences  $a_\ell(n)$  in the paper.

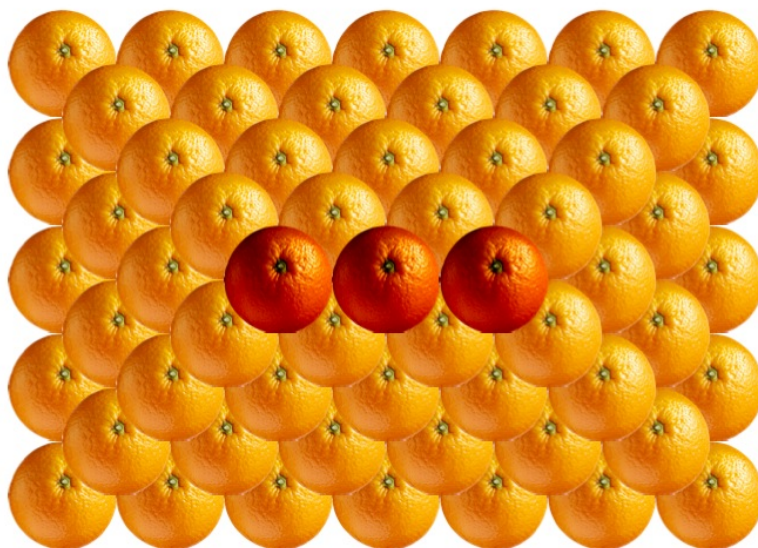


Figure 1: A stack with five layers of oranges and width  $\ell = 3$ .

In an alternative definition of the same problem in terms of height functions (as given in section 2), one observes that the local conditions on the height function are the same as those for plane and pyramid partitions. Propp in a post in the domino forum [1] in August 2014 asked whether one can find explicit formulae for the generating functions as is known in the case of plane and pyramid partitions [2, 3, 4]. The reformulation in terms of stacking oranges is due to R. Kenyon and the variant involving the number of oranges is due to Young [1].

In this paper, we address this issue by explicitly generating numbers for width  $a_\ell(n)$  for  $\ell = 1, 2, \dots, 6$  by adapting an algorithm due to Bratley-McKay [5]. We have been unable to find an explicit formula for the generating function. In the absence of a formula for the generating function, we address the following two questions in this paper.

1. For fixed  $n$ , what are the properties of  $a_\ell(n)$ ?
2. For fixed width  $\ell$ , what is the asymptotic behaviour of  $a_\ell(n)$ ?

The organisation of the the paper is as follows. After the introductory section where we state the problem at hand, in section 2, we give a formal definition of the problem and study the properties of  $a_\ell(n)$  for fixed  $n$ . We obtain an interesting conjecture for  $\ell \geq \lceil n/2 \rceil$ . In section 3, we first set upper and lower bounds on  $a_\ell(n)$  and numerically estimate the asymptotic behaviour using transition matrix Monte Carlo simulations for  $\ell \in [1, 6]$ . In section 4, we analyse the series of numbers obtained from exact enumeration to independently estimate the asymptotic behaviour as well as extrapolate the sequence of coefficients in order to obtain the next ten coefficients for  $a_1(n)$ . We conclude with a few remarks in section 5. Appendix A tabulates the results of our exact enumerations. In appendix B, we introduce a sub-class of plane partitions that appears naturally in this work and set bounds on the asymptotic behaviour of these restricted plane partitions.

## 2 Definitions and exact results

**Definition:** Let  $v = (x, y) \in \mathbb{Z}^2$  and for fixed  $\ell = 1, 2, 3, \dots$ , following [1] define

$$h_0^{(\ell)}(v) = \begin{cases} |x| + |y| & x < 0 \\ |x + y| & 0 \leq x < \ell \\ |y + \ell - 1| + |x - \ell + 1| & x \geq \ell \end{cases} .$$

The *height* function  $h$  on  $\mathbb{Z}^2$  is an integer-valued function that agrees with  $h_0^{(\ell)}$  almost everywhere (i.e., at all but finitely many places), is greater than or equal to  $h_0^{(\ell)}$  everywhere, and satisfies the condition that if  $u$  and  $v$  are adjacent locations in  $\mathbb{Z}^2$ ,  $|h(u) - h(v)| = 1$ . The last condition is called the *ice rule*.

**Definition:** Define the volume of the height function as follows:

$$n := \sum_{(x,y) \in \mathbb{Z}^2} \frac{1}{2} \left( h(x, y) - h_0^{(\ell)}(x, y) \right) . \quad (1)$$

**Definition:** Let  $a_\ell(n)$  denote the number of height functions with volume  $n$  for an initial configuration of width  $\ell$ .

### 2.1 The reduced height function

**Definition:** Define the reduced height function (on  $\mathbb{Z}^2$ ) as follows:

$$r(x, y) = \frac{1}{2} \left( h(x, y) - h_0^{(\ell)}(x, y) \right) , \quad (2)$$

where  $r(x, y)$  is a non-negative integer. Call the set of points  $(x, -x)$  (for  $0 \leq x < \ell$ ) where the topmost oranges lie, the *central crease*. The *northern crease*

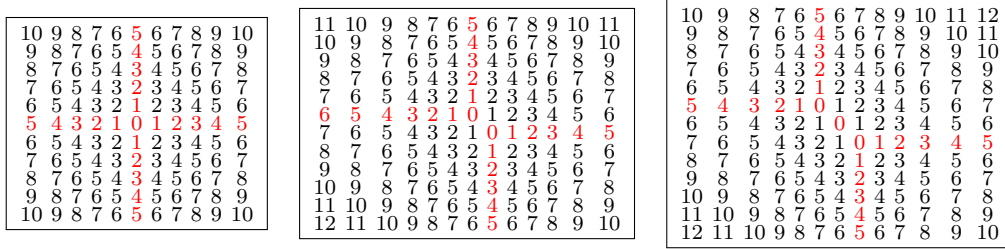


Figure 2: Initial height functions  $h_0^\ell$  for width  $\ell = 1, 2, 3$  inside a square. The red numbers partition the plane into four parts which we label as the NE, NW, SW and the SE parts. The creases are indicated in red.

is the set of points  $(0, y)$  with  $y > 0$  and the *western crease* is the set of points  $(x, 0)$  with  $x < 0$ . The *eastern crease* refers to the points  $(x + \ell - 1, \ell - 1)$  for  $x > 0$  and the *southern crease* to the set of points  $(\ell - 1, 1 - \ell + y)$  for  $y < 0$ . These points located on the creases are indicated in red numbers in the reference configurations shown in Figure 2.

**Proposition 2.1.** *The reduced height function is a weakly decreasing function as one moves away from the creases. Further, for unit steps along the N/S/E/W directions, it can change by at most one.*

*Proof.* Since the creases split configurations into four parts, we shall pick one part, say the NE part, and prove this property. In the NE part, going away from the crease corresponds to increasing the  $x$  or  $y$  coordinate by one. Consider a pair of neighbouring points,  $u = (x, y)$  and  $v = (x + 1, y)$ . Since  $h_0^{(\ell)}(v) - h_0^{(\ell)}(u) = 1$ , one has

$$r(u) - r(v) = \frac{1}{2}(h(u) - h(v) - h_0^{(\ell)}(u) + h_0^{(\ell)}(v)) = \frac{1}{2}(h(u) - h(v) + 1) .$$

Since  $|h(v) - h(u)| = 1$ , we see that  $(r(v) - r(u))$  is either 0 or  $-1$ . A similar proof shows that this is true for all other cases as well.  $\square$

Thus, given a configuration with volume  $n$ , it can be broken up into 2 plane partitions and 2 skew plane partitions with volumes  $(n_1, n_2, n_3, n_4)$  where  $\sum_{j=1}^4 n_j = n$ . These plane partitions are not the most general ones as the height condition is stronger than the weakly decreasing condition imposed for plane partitions (see Appendix B). We illustrate this split in Figure 3 for a random configuration with  $\ell = 6$  and volume = 120.

## 2.2 Exact enumeration

One would like to ask if there is a simple formula for  $a_\ell(n)$  or for its generating function. The first few numbers for width  $\ell \leq 5$  were computed by Ben

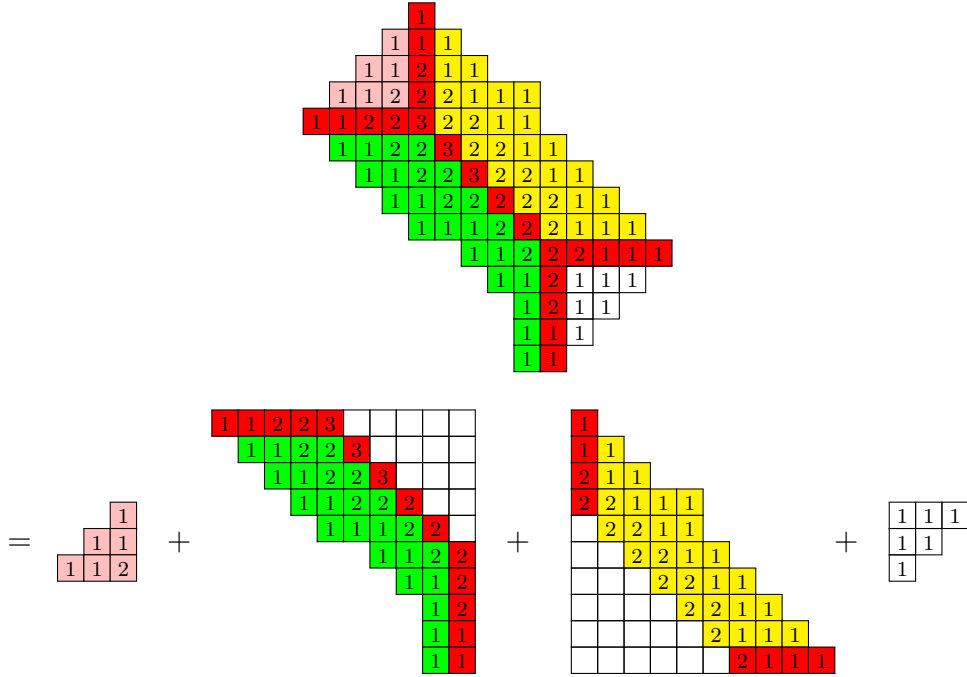
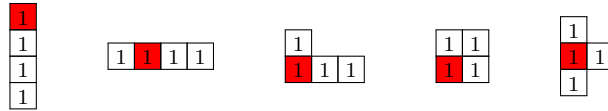


Figure 3: A random configuration of reduced height function for  $\ell = 6$  and volume 120. It is split into two PP's and two skew PP's.

Young and posted in the domino forum [1]. We adapted an algorithm due to Bratley and McKay to directly enumerate  $a_\ell(n)$ . Our initial numbers agree with Young's enumeration. Table 3 in Appendix A provides the the results of our exact enumeration of  $a_\ell(n)$  for widths  $\ell = 1$  to  $\ell = 6$ .

### 2.2.1 The $\ell = 1$ counting

There is a natural action of the dihedral group,  $D_8$ , that is generated by a rotation by  $\frac{\pi}{2}$  and a reflection  $(x, y) \rightarrow (-x, y)$  in the  $xy$ -plane. Below we indicate all possible configurations with fixed volume  $n = 4$  up to an overall action of  $D_8$ . Every point in  $\mathbb{Z}^2$  is represented by a square whose entry is the reduced height at the point. The red square is the origin with the horizontal line the  $x$ -axis and the vertical line the  $y$ -axis.



The multiplicities of the above configurations, (from left to right), under the action of  $D_8$  are 4, 4, 8, 4, 4 respectively. Thus there are 24 configurations with volume equal to 4. We are interesting in counting the number of configurations

with fixed volume  $n$ . Let  $a_1(n)$  denote the number of such configurations. The first few numbers are

$$1, 4, 10, 24, 51, 109, 222, 452, 890, 1732, 3298, 6204, 11470, 20970, 37842, 67572, \dots$$

Let  $A_\ell(q) = 1 + \sum_{m=1}^{\infty} a_\ell(m) q^m$  denote the generating function of the series  $a_\ell(m)$ , for fixed  $\ell$ . For  $\ell = 1$ , one has

$$A_1(q) := 1 + \sum_{n=1}^{\infty} a_1(n) q^n = 1 + q + 4q^2 + 10q^3 + 24q^4 + \dots, \quad (3)$$

$$= \prod_{m=1}^{\infty} (1 - q^m)^{-b_1(m)}, \quad (4)$$

where the second line defines  $b_1(m)$  for  $m = 1, 2, \dots$ . We have determined  $b_1(m)$  for  $m \leq 60$ . The first few numbers are:

$$1, 3, 6, 8, 9, 3, 2, 5, 28, 63, 86, 39, -\mathbf{112}, -\mathbf{303}, -\mathbf{326}, 109, 1020, 1725, 818, \dots \quad (5)$$

If all  $b_1(m) \geq 0$ , then one can look for a combinatorial problem that determines  $b_1(m)$ , thereby determining  $A_1(q)$ . However, we see that  $b_1(m)$  is not always positive – the negative terms have been shown in boldface above. This behaviour is similar to what happens for solid partitions where the analog of  $b_1(m)$  also oscillates between positive and negative values. We suspect that there might be **no** simple formula for the generating function. A similar situation holds for widths  $\ell > 1$ .

### 2.3 Studying $a_\ell(n)$ for fixed values of $n$

Given that there is no known analytical formula for the generating function, we next study the situation when  $n$ , the number of removed oranges, is kept fixed and study the properties as a function of  $\ell$ . Using exact data, we find that the following formulae appear to hold for  $\ell \geq \lceil n/2 \rceil$ . We set  $a_\ell(0) \equiv 1 = \binom{\ell}{0}$ . Using code which, for fixed  $\ell$ , generates the first few numbers in  $a_\ell(n)$ , enables us to

conjecture the following using fits to the data:

$$\begin{aligned}
a_\ell(2) &= \binom{\ell}{2} + 4 , \\
a_\ell(3) &= \binom{\ell}{3} + 6\ell \text{ for } \ell \geq 2 \\
a_\ell(4) &= \binom{\ell}{4} + 8\binom{\ell}{2} - \ell + 23 \text{ for } \ell \geq 2 , \\
a_\ell(5) &= \binom{\ell}{5} + 10\binom{\ell}{3} - 2\binom{\ell}{2} + 36\ell - 14 \text{ for } \ell \geq 3 , \\
a_\ell(6) &= \binom{\ell}{6} + 12\binom{\ell}{4} - 3\binom{\ell}{3} + 53\binom{\ell}{2} - 25\ell + 132 \text{ for } \ell \geq 3 , \\
a_\ell(7) &= \binom{\ell}{7} + 14\binom{\ell}{5} - 4\binom{\ell}{4} + 74\binom{\ell}{3} - 40\binom{\ell}{2} + 220\ell - 182 \text{ for } \ell \geq 4 , \\
a_\ell(8) &= \binom{\ell}{8} + 16\binom{\ell}{6} - 5\binom{\ell}{5} + 99\binom{\ell}{4} - 59\binom{\ell}{3} + 345\binom{\ell}{2} - 308\ell + 858 \text{ for } \ell \geq 4 , \\
a_\ell(9) &= \binom{\ell}{9} + 18\binom{\ell}{7} - 6\binom{\ell}{6} + 128\binom{\ell}{5} - 82\binom{\ell}{4} + 515\binom{\ell}{3} - 488\binom{\ell}{2} + 1463\ell - 1764 \text{ for } \ell \geq 5 .
\end{aligned}$$

For  $n = 2, 3, 4$ , the formulae have been proved [6]. The counting is fairly elaborate and does not reflect the simplicity of the above formulae. It hints at the existence of a statistic that refines  $a_\ell(n)$  but we have been unable to find one. The naïve guess that it counts the number of layers affected by a given configuration does not work. For  $5 \leq n \leq 9$ , the above formulae have been checked to be consistent with exact numbers given in Table 2 for  $\ell \leq 20$ . Observing their pattern, we conjecture that the following statement holds.

**Conjecture 2.2.** *For fixed  $n$  and  $\ell \geq \lceil n/2 \rceil$ ,  $a_\ell(n)$  is a polynomial of degree  $n$  in  $\ell$  such that*

$$a_\ell(n) = \sum_{k=0}^{\infty} g_k(n) \binom{\ell}{n-k} , \quad (6)$$

where  $g_k(x)$  is a polynomial of degree  $\lfloor \frac{k}{2} \rfloor$  in  $x$ .

The first nine values of  $n$  enables us to determine some of the polynomials to be as follows:

$$\begin{aligned}
a_\ell(n) &= \binom{\ell}{n} + 2n \binom{\ell}{n-2} - (n-3) \binom{\ell}{n-3} + (2n^2 - 5n + 11) \binom{\ell}{n-4} \\
&\quad - (2n^2 - 11n + 19) \binom{\ell}{n-5} + \frac{1}{6}(8n^3 - 57n^2 + 253n - 402) \binom{\ell}{n-6} + \cdots , \quad (7)
\end{aligned}$$

with  $\binom{\ell}{x} = 0$  for  $x < 0$ .

### 3 Asymptotics of $a_\ell(n)$

As we have seen, it appears that we cannot come up with a simple formula for the generating function for  $a_\ell(n)$ . With this in mind, we study their behaviour at large  $n$ , keeping the width  $\ell$  fixed. We first establish that for  $\ell \ll n^{1/3}$  and  $n \rightarrow \infty$  that  $n^{-2/3} \log a_\ell(n)$  is bounded. The proof follows a method similar to the one used to bound higher dimensional partitions [7]. We then use Monte Carlo simulations to study the asymptotic behaviour more precisely.

#### 3.1 Bounds on $a_\ell(n)$ for fixed $\ell$

**Proposition 3.1.** *For  $n \geq 2$ , the inequality,  $a_\ell(n) > a_\ell(n-1)$ , holds.*

*Proof.* Pick a configuration,  $\lambda$ , with volume  $n$  and let  $x > 0$  be the largest value of  $y$  such that  $r(y + \ell - 1, \ell - 1) = 1$ . If by setting  $r(x + \ell - 1, \ell - 1) = 0$ , we obtain a valid configuration with volume  $(n-1)$ , we say that  $\lambda$  has a removable 1-part located at  $(x, 0)$ . If  $\lambda$  has a removable 1-part, then setting  $r(x + \ell - 1, \ell - 1) = 0$  corresponds to removing the 1-part. For example, for  $\ell = 1$ ,  $\boxed{1} \boxed{1} \boxed{1} \boxed{1}$  has a removable 1-part at  $(2, 0)$  while  $\boxed{1} \boxed{1} \boxed{1} \boxed{1}$  has no removable 1-part. For  $n > 1$ , adding a 1-part to every configuration with volume  $(n-1)$  generates all configurations with volume  $n$  with a removable 1-part. Thus, one has

$$a_\ell(n) = a_\ell(n-1) + a_\ell(n|\text{no removable 1-part}) > a_\ell(n-1) \text{ for } n \geq 2. \quad (8)$$

**Remarks:** Given a configuration of volume  $(n-1)$ , it is always possible to add a removable one-part to create a unique configuration of volume  $n$  that has a removable one-part. For every  $n > 1$ , there exists at least one configuration without a removable one-part. Consider a configuration with  $r(x + \ell - 1, \ell - 1) = 0$  for all  $x > 0$  and  $r(\ell - 1, \ell - 1) = 1$ . (This proof has been adapted from a proof showing that  $p(n) > p(n-1)$ , where  $p(n)$  is the number of partitions of  $n$ , given in [8, see chap. 3].)  $\square$

**Proposition 3.2.** *As  $n \rightarrow \infty$  and  $\ell \ll n^{1/3}$ , one has  $\log a_\ell(n) > c_L n^{2/3}$  where  $c_L = 3^{2/3} \log 2 \approx 1.44$ .*

*Proof.* Consider the following special configuration with  $m$  layers (of oranges) completely removed. The  $(m+1)$ -th layer consists of  $n_0 = (m+1)(m+\ell)$  oranges that can all be removed independently of each other. By removing some or all of the oranges in the  $(m+1)$ -th layer, one creates  $2^{n_0}$  configurations with volume in the range  $[n - n_0, n]$  where

$$n = \sum_{k=1}^{m+1} k(k + \ell - 1) = \frac{1}{6}(1+m)(2+m)(2m+3\ell).$$



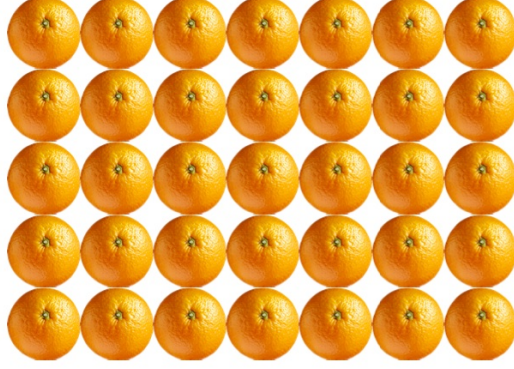


Figure 4: The fifth layer of a stack with  $\ell = 3$ .

We express  $m$  in terms of  $n$  by inverting the above expression to obtain

$$m = (3n)^{1/3} - \frac{(\ell + 2)}{2} + O(n^{-1/3}) .$$

Similarly, we can see that  $n_0 = (3n)^{2/3} + (3n)^{1/3} + O(1)$ . Since these  $2^{n_0}$  configurations do not exhaust all possible configurations, one has

$$\sum_{n'=n-n_0}^n a_\ell(n') > 2^{n_0} .$$

Since  $a_\ell(n' + 1) > a_\ell(n')$  for  $n' > 1$  from Proposition 3.1, we obtain

$$n_0 a_\ell(n) > \sum_{n'=n-n_0}^n a_\ell(n') > 2^{n_0} .$$

We thus get the following lower bound

$$\begin{aligned} \log a_\ell(n) &> (\log 2) n_0 - \log n_0 \\ &> (\log 2) n_0 = (\log 2)(3n)^{2/3} + O(n^{1/3}) =: c_L n^{2/3} + O(n^{1/3}) , \end{aligned} \quad (9)$$

with  $c_L = 3^{2/3} \log 2 \approx 1.4418$ . □

**Proposition 3.3.** *As  $n \rightarrow \infty$  and  $\ell \ll n^{1/3}$ , one has  $\log a_\ell(n) < c_U n^{2/3}$  where  $c_U = 3\zeta(3)^{1/3} \approx 3.1898$ .*

*Proof.* Let  $p_2(n)$  denote the number of plane partitions of  $n$  and  $\hat{p}_2^{(\ell)}(n)$  denote the number of skew plane partitions of shape  $\lambda/\mu_\ell$ , where  $\mu_\ell$  is the Ferrers diagram for partition  $(\ell, \ell - 1, \dots, 1)$  and  $\lambda$  the Ferrers diagram of a partition containing

$\mu_\ell$ . We obtain the following upper-bound for  $n \gg 1$ .

$$\begin{aligned} a_\ell(n) &< \sum_{\substack{n_i \in \mathbb{Z}_+ \\ \sum_i n_i = n}} \prod_{j=1}^2 p_2(n_j) \prod_{j=3}^4 \hat{p}_2^{(\ell)}(n_j) , \\ &< \sum_{\substack{n_i \in \mathbb{Z}_+ \\ \sum_i n_i = n}} \prod_{j=1}^2 p_2(n_j) \prod_{j=3}^4 p_2(n_j + \frac{\ell^2}{2}(3n)^{1/3}) , \end{aligned}$$

where in the second line, we have replaced the counting of skew plane partitions to plane partitions by filling in  $\mu_\ell$  with the largest possible value which can be estimated to be  $(3n)^{1/3}$ . Since  $\ell \ll n^{1/3}$ , we assume that it is  $O(1)$ . Since  $p_2(n)$  is a monotonically increasing function of  $n$ , it follows that among all partitions of  $n$  into four parts, the largest term in the above product occurs when all  $n_i$  are equal. Thus one has  $\ell^2 n^{1/3} \ll n_j$  for  $j = 3, 4$ . Using this, we obtain

$$a_\ell(n) < p(n|4 \text{ parts}) p_2\left(\frac{n}{4}\right)^4 . \quad (10)$$

where  $p(x|4 \text{ parts}) = O(x^3)$  is the number of partitions of  $x$  into four parts. Taking logarithms and discarding terms that grow as  $\log n$  that arise from  $p(n|4 \text{ parts})$ , we obtain

$$\log a_\ell(n) < 4 \log p_2\left(\frac{n}{4}\right) \sim 3\zeta(3)^{1/3} n^{2/3} = 3.1898 n^{2/3} , \quad (11)$$

on using  $\log p_2(n) \sim \frac{3}{2}(2\zeta(3))^{1/3} n^{2/3}$ , see [9].  $\square$

Combining our lower and upper bounds, we obtain the following bounds:

$$\boxed{3^{2/3} \log 2 < n^{-2/3} \log a_\ell(n) < 3\zeta(3)^{1/3}} . \quad (12)$$

This suggests that  $n^{-2/3} \log a_\ell(n) \rightarrow \text{constant}$  as  $n \rightarrow \infty$ .

**Conjecture 3.4.** *For  $\ell \ll n^{1/3}$ ,  $n^{-2/3} \log a_\ell(n) \sim$  an  $\ell$ -independent constant as  $n \rightarrow \infty$ .*

A heuristic proof of  $\ell$ -independence is as follows. Since  $\ell \ll n^{1/3}$ , arguments similar to those that lead to the lower bound show that a generic random configuration will be a rectangle of side  $(3n)^{1/3}[(3n)^{1/3} + \ell] \sim (3n)^{2/3} + \ell O(n^{1/3})$ . This suggests that the  $\ell$ -dependence is suppressed by at least a power of  $n^{1/3}$ . We shall provide evidence for this using Monte Carlo simulations to estimate the constant for  $\ell = 1, \dots, 6$ .

### 3.2 Studying asymptotics using Monte Carlo simulations

Let  $\lambda$  denote a particular height function (or equivalently a stack of oranges) with volume  $n$ . We indicate this by  $\lambda \vdash n$ . Let  $n_+(\lambda)$  ( $n_-(\lambda)$ ) denote the number of oranges that can be removed (resp. added) to obtain a valid height function with volume  $(n+1)$  (resp.  $(n-1)$ ). Define  $N_{\pm}(n)$  as follows:

$$\begin{aligned} N_+(n) &:= \frac{\sum_{\lambda \vdash n} n_+(\lambda)}{\sum_{\lambda \vdash n} 1} = \frac{\sum_{\lambda \vdash n} n_+(\lambda)}{a_{\ell}(n)} \quad \text{and} \\ N_-(n) &:= \frac{\sum_{\lambda \vdash n} n_-(\lambda)}{\sum_{\lambda \vdash n} 1} = \frac{\sum_{\lambda \vdash n} n_-(\lambda)}{a_{\ell}(n)}, \end{aligned} \quad (13)$$

where the sums run over all height functions with volume  $n$ . For  $n > 1$ , one has the identity

$$N_+(n-1) a_{\ell}(n-1) = N_-(n) a_{\ell}(n). \quad (14)$$

Given  $N_+(n)$  and  $N_-(n)$ , one can determine  $a_{\ell}(n)$  by recursively using the formula and using  $a_{\ell}(0) = 1$ . That is,

$$a_{\ell}(n) = \prod_{m=0}^{n-1} \frac{N_+(m)}{N_-(m+1)}, \quad (15)$$

or for  $n > n_0$  (where  $a_{\ell}(n_0)$  has been exactly enumerated)

$$a_{\ell}(n) = \prod_{m=n_0}^{n-1} \frac{N_+(m)}{N_-(m+1)} a_{\ell}(n_0), \quad (16)$$

The transition matrix Monte Carlo simulation we use estimates averages for  $N_{\pm}(n)$  for  $n \in [1, 4100]$  for  $\ell = 1, \dots, 6$ . We assume that  $\log a_{\ell}(n)$  takes the following asymptotic form:

$$\log a_{\ell}(n) \sim c_0 n^{2/3} + c_1 \log n + c_2 + c_3 n^{1/3}. \quad (17)$$

Using this form, one can show that

$$\log \frac{a_{\ell}(n)}{a_{\ell}(n-1)} = \log \frac{N_+(n-1)}{N_-(n)} \sim \frac{2}{3} c_0 n^{-1/3} + c_1 n^{-1} + \frac{1}{3} c_3 n^{-2/3}. \quad (18)$$

For our Monte Carlo fits, we use a variant of the above formula

$$\log \frac{a_{\ell}(n)}{a_{\ell}(n-1)} \sim \left(\frac{2}{3} + \frac{1}{9n}\right) c_0 n^{-1/3} + c_1 n^{-1} + \left(\frac{1}{3} + \frac{1}{9n}\right) c_3 n^{-2/3}, \quad (19)$$

where we have added some sub-leading terms (suppressed by  $1/n$ ) without changing the number of parameters. This formula is suited to our Monte Carlo simulation as it relates the quantities computed in the simulation to the parameters that

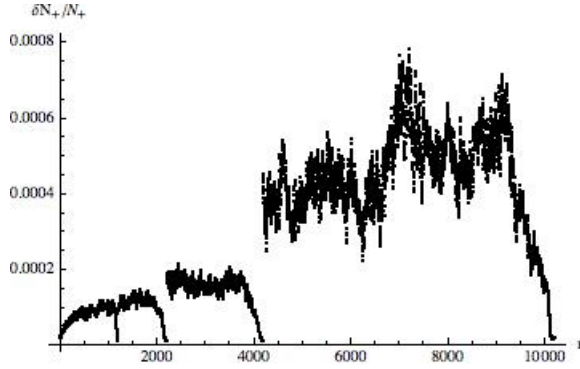


Figure 5: Plot of statistical error,  $\frac{\delta N_+}{N_+}$ , against  $n$ . The merger of four data sets is also clearly visible. All statistical errors show similar behaviour.

appear in the asymptotic form for  $a_\ell(n)$ . The parameter  $c_2$  has to be determined separately as it drops out of the above formula.

The Monte Carlo simulation is a randomisation of the Bratley-McKay algorithm. We adapted the Transition Matrix Monte Carlo method described in [12] to study solid partitions restricted to be in a box and to estimate the asymptotics of solid partitions in [11]. As in those papers, we use a fictitious temperature to get a wider coverage for values of  $n \in [1, N_{\max}]$ . The averages for estimating  $N_\pm(n)$  are carried out at infinite temperature. We carried out several runs with different values of  $N_{\max} = 1200, 2200, 4200, 10200$ . For each value of  $N_{\max}$ , we carried out runs with distinct seeds for the random number generator in order to get an estimate of the statistical error in  $N_\pm(n)$ . The numbers from all runs were then combined into a single data set with statistical errors. For  $n \in [1, 30]$ , the values of  $N_\pm(n)$  were compared with exact values (again computed using the Bratley-McKay algorithm [6]) to see if the statistical errors that we obtained were consistent with actual ones. The exact numbers also enabled us to establish that longer runs lead to lower statistical errors. As a proof of concept, we also verified that a similar randomisation of the Bratley-McKay code for ordinary partitions worked. The runs with  $N_{\max} = 10200$  were not used in any of our fits as their errors were too large and were only used to verify that our fits do reproduce the asymptotic behaviour correctly.

### 3.2.1 Summary of Monte Carlo results

We carried out three sets of fits using estimates for  $N_\pm(n)$  for values of  $n$  in the range  $[\ell^3 + 10, 4100]$ . The data for  $n > 4100$  has larger errors and hence is only used to see if the trends of the fits are consistent.

**Fit 1:** This is the formula given in Eq. (19) which involves three parameters. The first fit gives

$\ell$	$c_0$	$c_3$	$c_1$
1	2.34426	-0.0110902	-0.746477
2	2.34437	-0.0156179	-0.740064
3	2.34441	-0.0281878	-0.670066
4	2.34492	-0.0669717	-0.5053
5	2.34558	-0.119494	-0.248828
6	2.34538	-0.144212	0.0401944

**Fit 2:** The second fit is one where a fourth parameter is introduced by adding a term  $\varepsilon n^{-4/3}$  to the right hand side of Eq. (19). The second fit gives

$\ell$	$c_0$	$c_3$	$c_1$	$\varepsilon$
1	2.34401	0.0028884	-0.78056	0.0630788
2	2.34417	0.00278436	-0.770915	0.064166
3	2.3379	0.0104204	-0.783435	0.277578
4	2.34397	-0.00361808	-0.712328	0.589102
5	2.34444	-0.0387065	-0.538716	0.935576
6	2.34329	-0.0140978	-0.575709	2.2077

**Fit 3:** A third form for the asymptotic behaviour, based on the (leading) singularity of the generating function, is

$$a_\ell(n) \sim A \mu^{n^{2/3}} n^g .$$

Comparing with the first asymptotic formula, we see that  $A = e^{c_2}$ ,  $\mu = e^{c_0}$ ,  $g = c_1$  and  $c_3 = 0$ . For the third fit we also added the term  $\varepsilon n^{-4/3}$  term, giving

$\ell$	$c_0$	$\mu = e^{c_0}$	$g = c_1$	$\varepsilon$
1	2.34407	10.4236	-0.7741296	0.0520715
2	2.34412	10.4241	-0.777704	0.0773793
3	2.34397	10.4225	-0.754525	0.210345
4	2.34391	10.4219	-0.723663	0.620114
5	2.34389	10.4217	-0.673265	1.35694
6	2.34348	10.4174	-0.522174	2.02021

We see that forcing  $c_3 = 0$  makes the value of  $c_0$  almost independent of  $\ell$  providing evidence to our conjecture that  $c_0$  is  $\ell$ -independent. We assign it the  $\ell$ -independent value

$$\boxed{c_0 = 2.344 \pm 0.001 \text{ or } \mu = 10.42 \pm 0.01 .} \quad (20)$$

The errors here are crude estimates based on comparing how the numbers change when compared to the second fit. Further the parameter  $g = c_1$  is clearly  $\ell$ -dependent.

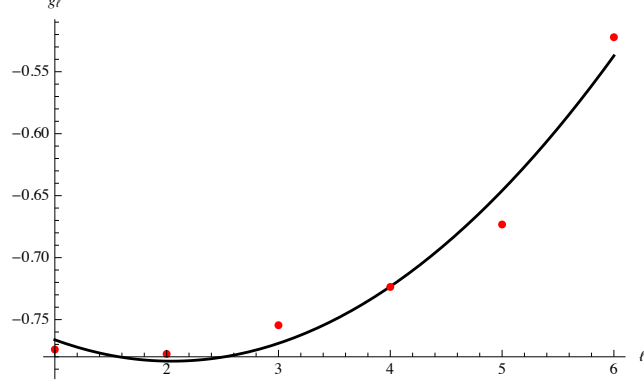


Figure 6:  $g_\ell$  vs  $\ell$  along with a quadratic fit which gives  $g_\ell = -0.717604 - 0.0644484\ell + 0.0157537\ell^2$ .

The main conclusion that we can draw from the Monte Carlo simulations is that the asymptotic behaviour of  $a_\ell(n)$  is consistent with the following form:

$$a_\ell(n) \sim A_\ell \mu^{n^{2/3}} n^{g_\ell} = e^{c_\ell^2} \mu^{n^{2/3}} n^{g_\ell} , \quad (21)$$

where  $\mu = 10.42 \pm 0.01$  is an  $\ell$ -independent constant and  $A_\ell = e^{c_\ell^2}$  and  $g_\ell$  are  $\ell$ -dependent constants.

We still need to estimate  $A_\ell$  or equivalently the constant  $\alpha_3^\ell$  as it does not appear in the fits based on Eq. (19). We need explicit values for  $a_\ell(n)$  – this is something we indirectly determine using our estimates for  $N_\pm(n)$  combined with Eq. (16) with  $n_0$  chosen to be the largest possible value appearing in our explicit enumeration given in Table 3. We fit to the formula

$$n^{-2/3} \log a_\ell(n) \sim c_0 + c_1 n^{-2/3} \log n + c_2 n^{-2/3} - 3\varepsilon n^{-1} , \quad (22)$$

with the values of  $c_0$ ,  $g_\ell$  and  $\varepsilon$  determined by Fit 3. We use small values of  $n \in [\max(10, \ell^3), \ell^3 + 100]$  as it is here that this term contributes significantly and statistical errors are small.

$\ell$	$c_2$	$A = e^{c_2^2}$
1	-1.55101	0.212034
2	-1.2617	0.283173
3	-0.64815	0.523012
4	0.356079	1.42772
5	1.64144	5.16257
6	2.52126	12.4442

## 4 Series analysis of partition-type series

Much of the pre-existing work on methods to extract the asymptotic form of coefficients numerically from a finite number of coefficients assumes the form

$$a_n \sim \text{const.} \mu^n n^g,$$

with corresponding generating function

$$\sum a_n x^n \sim \text{const.} (1 - \mu x)^{-1+g}.$$

Many problems in enumerative combinatorics and statistical mechanics have such singularities. Methods for the analysis of coefficients in order to estimate the growth constant  $\mu$ , the exponent  $g$  and the amplitude, given by the constant pre-multiplier have been well-developed over the past few decades, and are discussed in [13].

In contrast, for the type of asymptotics associated with plane partitions and related series, the literature is very scant indeed. Accordingly, we first take a known problem, the asymptotics of plane partitions, and develop appropriate methods of series analysis. We then apply these methods to the problem at hand, the square-ice analogue of plane partitions.

### 4.1 Analysis of plane-partition series

The generating function of plane partitions, due to MacMahon [2], is well-known and is given by

$$P(x) = \sum p_n x^n = \prod_{k \geq 0} \frac{1}{(1 - x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + \dots$$

The asymptotics are also well-studied, and are given by [9, 10]

$$n^{-2/3} \log p_n \sim c_0 + c_1 \frac{\log n}{n^{2/3}} + \frac{c_2}{n^{2/3}} + O(n^{-4/3}), \quad (23)$$

where  $c_0 = 2.00945 \dots$ ,  $c_1 = -\frac{25}{36} = -0.694444 \dots$ , and  $c_2 = -1.4631 \dots$ .

It is straightforward to generate as many terms as required from the generating function. We have chosen to generate 200 terms, and investigate the assumed form

$$\tilde{p}_n = n^{-2/3} \log p_n \sim c_0 + c_1 \frac{\log n}{n^\alpha} + \frac{c_2}{n^\alpha},$$

with higher order terms neglected. That is to say, we assume ignorance of the exponent  $\alpha$ , and set out to estimate its value.

Forming first-differences, so that

$$s_n = \tilde{p}_n - \tilde{p}_{n-1} \sim -c_1 \alpha \frac{\log n}{n^{1+\alpha}} + O\left(\frac{1}{n^{1+\alpha}}\right),$$

then a plot of  $s_n$  against  $\frac{\log n}{n^{(1+\alpha)}}$  should be linear for the “correct” choice of  $\alpha$  and  $n$  sufficiently large. This is not a particularly sensitive test, but one might expect to establish if  $\alpha$  is closer to 1 or to zero. In Figure 8 we show such a plot for three values of  $\alpha$ . For  $\alpha = 1$ , shown at left, the plot is slightly convex, while the right-most plot, corresponding to  $\alpha = 0.5$  is significantly concave, while the central plot, corresponding to  $\alpha = 0.75$  is essentially linear. The correct value of  $\alpha$  is of course  $2/3$  in this case.

An alternative way to estimate  $\alpha$  is to plot  $\log\left(\frac{s_n}{\log n}\right)$  against  $\log n$ . This should have gradient  $-(1 + \alpha)$ . This plot (not shown) is indeed visually linear. If one calculates the local gradient, defined as the gradient of successive pairs of points, one sees a steady variation with  $n$ . This local gradient is plotted against  $n^{-2/3}$  in the left-most plot in figure 9. It is clear that this is extrapolating to a value around  $-1.68$  as  $n \rightarrow \infty$ , which is quite close to the known exact value  $-5/3$ .

Assuming we have found the value of  $\alpha$  correctly to be  $2/3$ , we are now in a position to estimate the constants appearing in the asymptotic expression (23). We fit successive triples of terms  $\{\tilde{p}_{n-2}, \tilde{p}_n, \tilde{p}_{n+2}\}$  in order to estimate the constants  $\{c_0(n), c_1(n), c_2(n)\}$ . (Alternate terms are used to reduce an odd-even effect that would otherwise cause oscillatory estimates). We show the estimates of these constants, plotted against  $n^{-4/3}$ ,  $n^{-2/3}$  and  $n^{-1/3}$  respectively in figures 9 and 10 below. The estimates of  $c_0$  are clearly going to a value around 2.0095, which is very close to the exact value. The estimates of  $c_1$  appear to be going to a limit around  $-0.695$ , in good agreement with the known exact value,  $-0.69444 \dots$ . The estimate for  $c_2 \approx -1.436$  which is comparable to the known value of  $c_2 = -1.4631 \dots$ .

We have repeated the above analysis with an additional term  $c_3/n^{1/3}$  in (23), and the estimators of  $c_3$  are clearly going to a value close to 0, consistent with the absence of such a term.

## 4.2 Analysis of square-ice series

We now repeat the above analysis for the sequence  $a_1(n)$  which is known exactly for  $n \leq 60$ . We have recently developed a numerical technique that allows one to approximately extend a given series by several coefficients, with a level of precision that is good enough for this type of graphical analysis, see [14]. In this way we have extended the series by 10 further terms, and these are quoted in Table 1 alongside the estimates from the Monte Carlo simulations.

As in the preceding case, we first form the sequence

$$\tilde{p}_n = n^{-2/3} \log a_1(n) \sim c_0 + c_1 \frac{\log n}{n^\alpha} + \frac{c_2}{n^\alpha},$$

with higher order terms neglected, and we calculate the first-differences,  $s_n = \tilde{p}_n - \tilde{p}_{n-1}$  and plot  $s_n$  against  $\frac{\log n}{n^{(1+\alpha)}}$ . We show the results in Figure 11, again for three



values of  $\alpha$ . The situation is exactly the same as for plane partitions. For  $\alpha = 1$ , shown at left, the plot is slightly convex, while the right-most plot, corresponding to  $\alpha = 0.5$  is significantly concave, while the central plot, corresponding to  $\alpha = 0.75$  is essentially linear. This suggests that the correct value of  $\alpha$  is also  $2/3$  in this case.

Estimating  $\alpha$  by plotting  $\log\left(\frac{s_n}{\log n}\right)$  against  $\log n$  again gives a visually linear plot. More interesting is the plot of the local gradient, and this is shown plotted against  $1/n^{2/3}$  in Figure 11. This appears to extrapolate to a value around  $-1.68$  as  $n \rightarrow \infty$ , just as for plane partitions, which again suggests that the correct exact value should be  $-5/3$ .

Assuming we have found the value of  $\alpha$  correctly to be  $2/3$ , we are now in a position to attempt to estimate the constants appearing in the asymptotic expression (23). As for the case with plane partitions, we fit successive triples of terms  $\{\tilde{p}_{n+2}, \tilde{p}_n, \tilde{p}_{n+2}\}$  in order to estimate the constants  $\{c_0(n), c_1(n), c_2(n)\}$ . We show the estimates of these constants, plotted against suitable powers of  $n$ ,  $\{n^{-4/3}, n^{-2/3}, n^{-1/3}\}$ , in Figures 12 and 13 below. All display oscillatory behaviour which makes extrapolation difficult, if not impossible. If we assume – and this is indeed a leap of faith, justifiable only because the results are consistent with the Monte Carlo analysis – that this oscillatory trend persists with decreasing amplitude, then we can estimate  $c_0 \approx 2.345$ ,  $c_1 \approx -0.75$  and  $c_2 \approx -1.7$ .

These results are entirely consistent with, though less accurate than, the Monte Carlo estimates obtained from the third fit, which assumes  $c_3$  is zero (that is, there is no term  $O(n^{-1/3})$  in Eq. (23)).

### 4.3 Behaviour of $a_\ell(n)$ for $\ell > 1$ .

From our Monte Carlo work we concluded in Eq. (21) that for  $\ell \gg n^{1/3}$

$$a_\ell(n) \sim A_\ell \mu^{n^{2/3}} n^{g_\ell}$$

where  $A_\ell$  and  $g_\ell$  are  $\ell$ -dependent, while  $\mu$  is not. For this investigation we can make the weaker assumption that the exponent  $2/3$  can be positive exponent  $\theta$ , as we will eliminate this dominant term. While our series analysis is not accurate enough to give a good estimate of  $g_\ell$  directly (as shown above), we instead focus on  $g_\ell - g_1$ . One has

$$\hat{a}_\ell(n) \equiv \frac{a_\ell(n)}{a_1(n)} \sim \frac{A_\ell}{A_1} n^{g_\ell - g_1}, \quad (24)$$

and so the exponent  $\hat{g}_\ell = g_\ell - g_1$  can be estimated from the ratios of successive terms  $\hat{a}_\ell(n)$ . That is to say,

$$r_\ell(n) \equiv \frac{\hat{a}_\ell(n)}{\hat{a}_\ell(n-1)} \sim 1 + \frac{\hat{g}_\ell}{n}. \quad (25)$$

$n$	$\text{sf}(n)$	$\text{mc}_1(n)$	% error $\text{mc}_1(n)$
61	$5.08349035674 \times 10^{13}$	50834979702073	0.00641281
62	$7.460434311 \times 10^{13}$	74604412596394	0.0128265
63	$1.092771318 \times 10^{14}$	109276600121877	0.0188305
64	$1.597623083 \times 10^{14}$	159761033617959	0.0245731
65	$2.3313927896 \times 10^{14}$	233136871953374	0.030269
66	$3.39600366875 \times 10^{14}$	339595034177620	0.0362506
67	$4.9379657155 \times 10^{14}$	493785801126495	0.0421327
68	$7.1674931 \times 10^{14}$	716736440905024	0.0478466
69	$1.0385930349 \times 10^{15}$	1038570180194263	0.053417
70	$1.5023341234 \times 10^{15}$	1502380905370668	0.0590596

Table 1: Comparing the series estimates,  $\text{sf}_1(n)$ , with the Monte Carlo estimates,  $\text{mc}_1(n)$ , for  $a_1(n)$ . The differences of the two estimates are consistently lower than the error in column 3 by an order of magnitude.

So a plot of  $r_\ell(n)$  against  $1/n$  should be linear, with slope  $\hat{g}_\ell$ , and with ordinate 1 as  $n \rightarrow \infty$ . We show in Figure 14 the ratios  $r_\ell(n)$  plotted against  $1/n$  for  $\ell = 6, 5, 4, 3, 2$  reading from top to bottom. It can be seen that these ratio plots are behaving as expected, but with a small amount of curvature due to the effect of unknown higher-order terms in (25). We attempt to accommodate these by calculating the local gradient

$$\hat{g}_\ell(n) = n(r_\ell(n) - 1) \sim \hat{g}_\ell + o(1).$$

In fact, it appears empirically that the term  $o(1)$  can be replaced by  $O(1/n)$ , as plots of  $\hat{g}_\ell(n)$  against  $1/n$  appear to be essentially linear. In this way we estimate

$$g_\ell \approx 0.0, 0.058, 0.17, 0.37, 0.64$$

for  $\ell = 2, 3, 4, 5, 6$  respectively. These differences lie somewhere between those obtained from fit 2 and fit 3 in our Monte Carlo analysis. Note that for  $\ell > 3$ , we do not have exact numbers for  $n > \ell^3$ , so the above analysis can be taken seriously only for  $\ell \leq 3$ . As the series analysis is independent of any assumptions except the form (21), we might expect series analysis to be more accurate for this parameter.

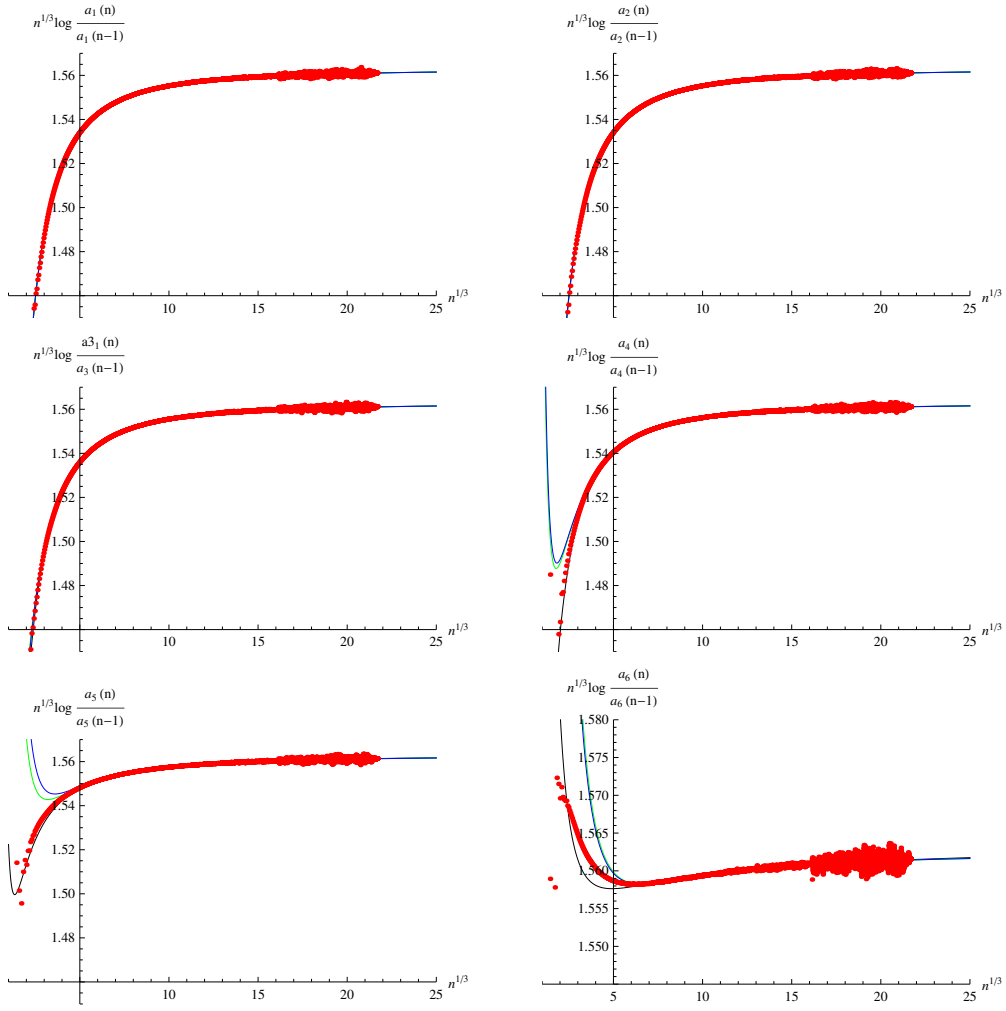


Figure 7: Plots of  $\log \frac{a_\ell(n)}{a_\ell(n-1)}$  vs  $n^{1/3}$  for  $\ell = 1, \dots, 6$ . The three fits are plotted with fit1 in black, fit2 in green, fit3 in blue and red for the Monte Carlo data. The fits should work only for  $n^{1/3} > \ell$  and the fits differ when  $n^{1/3} < \ell$ .

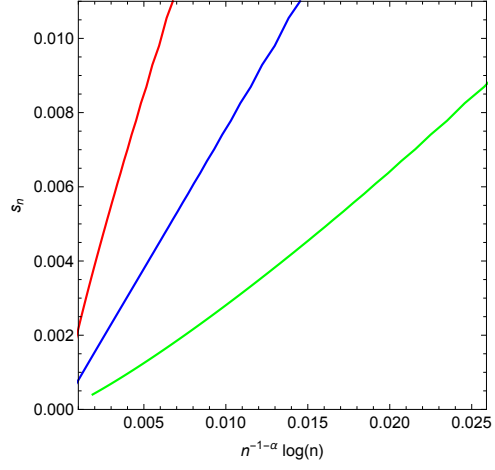


Figure 8: Left-most plot,  $\alpha = 1$ , central plot,  $\alpha = 0.75$  and right-most plot,  $\alpha = 0.5$ . for plane partitions.

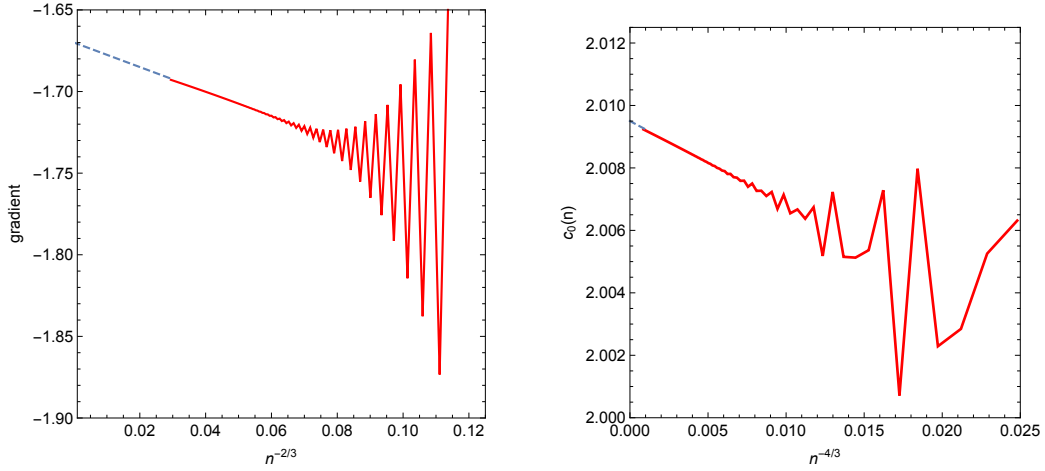


Figure 9: (Left) Plot of local gradient against  $n^{-2/3}$  and (right) Plot of  $c_0(n)$  against  $n^{-4/3}$  for plane partitions.

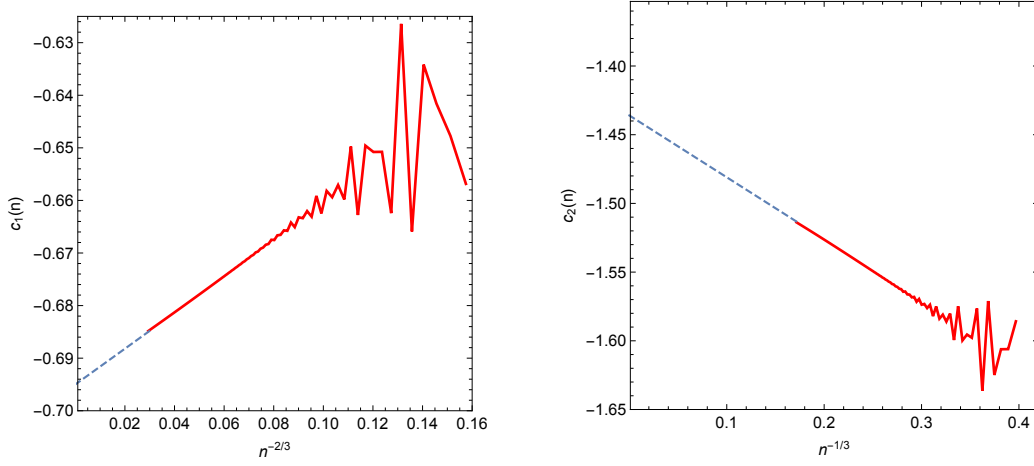


Figure 10: Plots of  $c_1(n)$  and  $c_2(n)$  against  $n^{-2/3}$  (resp.  $n^{-1/3}$ ) for plane partitions.

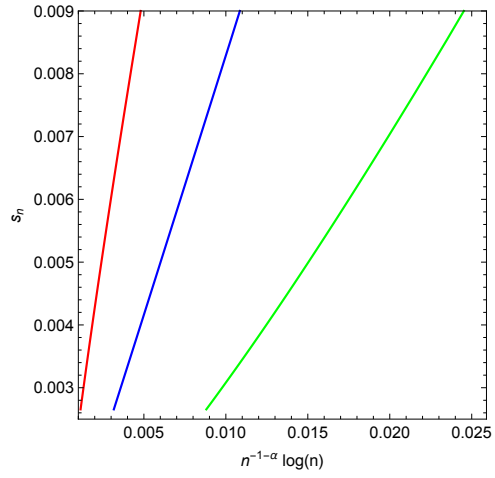


Figure 11: Left-most plot,  $\alpha = 1$ , central plot,  $\alpha = 0.75$  and right-most plot,  $\alpha = 0.5$ . for the square-ice series.

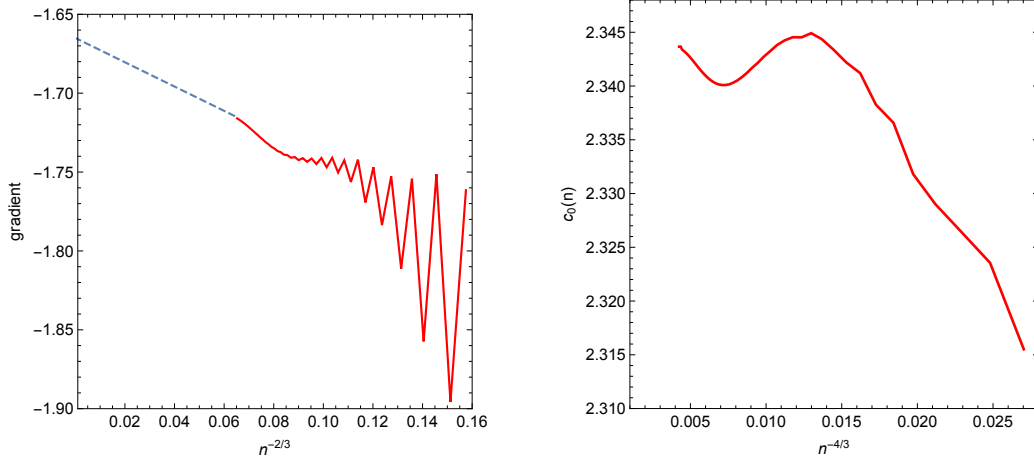


Figure 12: (Left) Plot of local gradient against  $n^{-2/3}$  and (right) plot of  $c_0(n)$  against  $n^{-4/3}$  for the square-ice series.

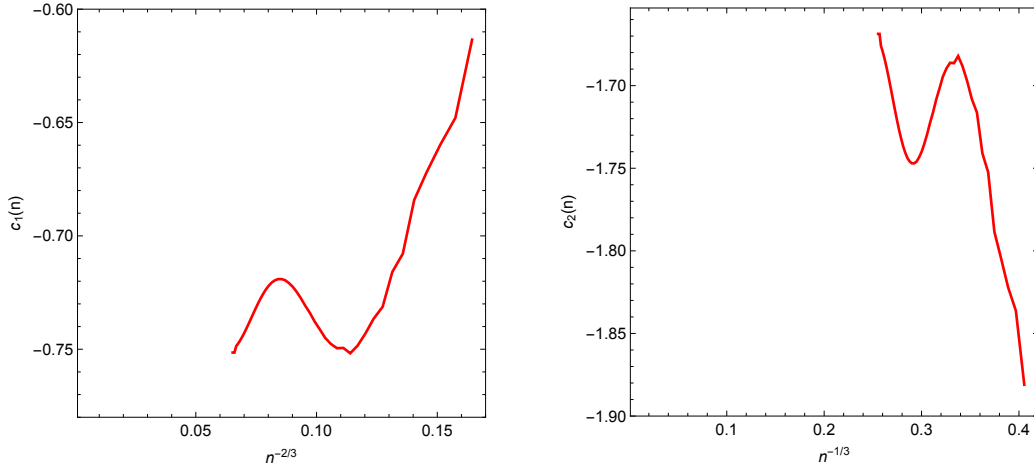


Figure 13: Plot of  $c_1(n)$  and  $c_2(n)$  against  $n^{-2/3}$  (resp.  $n^{-1/3}$ ) for the square ice series.

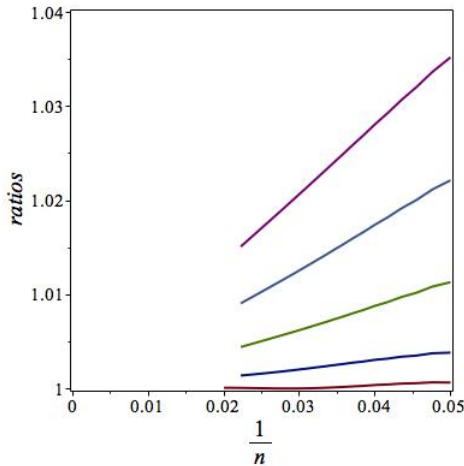


Figure 14: Plot of ratios  $r_\ell(n)$  against  $1/n$  for  $\ell = 6, 5, 4, 3, 2$  reading from top to bottom.

## 5 Concluding Remarks

In this paper, we have addressed several aspects of the square-ice analogue of plane partitions. Our exact enumerations have non-trivially extended the numbers provided by Young. As expected, the asymptotic behaviour is similar to that of plane partitions. We showed this by establishing the leading asymptotic behaviour and then using Monte Carlo simulations to provide evidence for behaviour of the form given in Eq. (21). The lack of a generating function makes it an ideal testing ground for the series extension methods that one of us (AG) has developed. In this context, our Monte Carlo simulations provide an independent check on the method.

The exact data as well as our Monte Carlo simulations clearly indicate that for  $n > 1$  and  $\ell > \ell'$ , one has  $a_\ell(n) > a_{\ell'}(n)$ . However, we have not proved this statement and will leave it as an open conjecture. Conjecture 2.2 also remains open and suggests the existence of a new statistic that might enable one to prove the conjecture. Of course, it remains to be seen if one can find explicit formulae for the generating functions for  $a_\ell(n)$ .

**Acknowledgments:** We thank Nicolas Destainville for useful conversations as well as sharing his Monte Carlo code for solid partitions. We are grateful to Jim Propp, Rick Kenyon, Ben Young and others members of the domino forum for drawing our attention to this problem.

## A Numbers from exact enumeration

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a_1(n)$	1	1	4	10	24	51	109	222	452	890	1732	3298	6204	11470	20970	37842
$a_2(n)$	1	2	5	12	29	64	139	286	582	1148	2227	4234	7950	14692	26842	48438
$a_3(n)$	1	3	7	19	44	98	213	448	918	1832	3584	6882	13012	24220	44480	80678
$a_4(n)$	1	4	10	28	68	158	350	750	1559	3170	6292	12252	23445	44164	81995	150288
$a_5(n)$	1	5	14	40	103	247	567	1252	2668	5539	11214	22247	43300	82871	156152	290202
$a_6(n)$	1	6	19	56	152	378	898	2042	4476	9526	19740	39978	79342	154650	296489	560022
$a_7(n)$	1	7	25	77	219	567	1392	3263	7354	16048	34055	70503	142842	283832	554196	1065070
$a_8(n)$	1	8	32	104	309	834	2116	5114	11849	26520	57620	121950	252256	511180	1016878	1989150
$a_9(n)$	1	9	40	138	428	1204	3159	7870	18747	43036	95729	207125	437402	903914	1831938	3647757
$a_{10}(n)$	1	10	49	180	583	1708	4637	11906	29158	68652	156336	345780	745450	1570920	3243407	6573672
$a_{11}(n)$	1	11	59	231	782	2384	6699	17726	44627	107763	251213	567936	1249864	2685688	5648561	11652141
$a_{12}(n)$	1	12	70	292	1034	3278	9534	25998	67276	166602	397542	918580	2063435	4520696	9684744	20332156
$a_{13}(n)$	1	13	82	364	1349	4445	13379	37596	99983	253894	620074	1464231	3357015	7498084	16360443	34952692
$a_{14}(n)$	1	14	95	448	1738	5950	18528	53650	146605	381704	954023	2302014	5386122	12263456	27250699	59239788
$a_{15}(n)$	1	15	109	545	2213	7869	25342	75605	212253	566525	1448904	3572062	8528301	19792143	44785054	99055075
$a_{16}(n)$	1	16	124	656	2787	10290	34260	105290	303628	830660	2173572	5474290	13335035	31540550	72667194	163510356
$a_{17}(n)$	1	17	140	782	3474	13314	45811	144998	429428	1203961	3222775	8290859	20603148	49659502	116480131	266609605
$a_{18}(n)$	1	18	157	924	4289	17056	60627	197578	600837	1725998	4725599	12415980	31472081	77292052	184550272	429643458
$a_{19}(n)$	1	19	175	1083	5248	21646	79457	266540	832108	2448742	6856258	18395107	47555212	118985326	289169418	684647303
$a_{20}(n)$	1	20	194	1260	6368	27230	103182	356174	1141253	3439858	9847768	26976044	71115601	181255026	448307314	1079349306

Table 2: Numbers in red are those for which the (generic) formula for  $a_\ell(n)$  is anticipated to fail. The numbers have been checked for  $n \leq 9$  and  $\ell \leq 20$ .



$n$	$a_1(n)$	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$	$a_6(n)$
1	1	2	3	4	5	6
2	4	5	7	10	14	19
3	10	12	19	28	40	56
4	24	29	44	68	103	152
5	51	64	98	158	247	378
6	109	139	213	350	567	898
7	222	286	448	750	1252	2042
8	452	582	918	1559	2668	4476
9	890	1148	1832	3170	5539	9526
10	1732	2227	3584	6292	11214	19740
11	3298	4234	6882	12252	22247	39978
12	6204	7950	13012	23445	43300	79342
13	11470	14692	24220	44164	82871	154650
14	20970	26842	44480	81995	156152	296489
15	37842	48438	80678	150288	290202	560022
16	67572	86509	144697	272150	532430	1043404
17	119368	152902	256775	487388	965395	1919708
18	208943	267783	451305	863887	1731351	3491081
19	362389	464766	786008	1516592	3073660	6280514
20	623438	800095	1357414	2638648	5404984	11185375
21	1064061	1366512	2325540	4552488	9420512	19734004
22	1802976	2316840	3954366	7792566	16282463	34509347
23	3033711	3900502	6676369	13239698	27922063	59847208
24	5071418	6523432	11196599	22336630	47527430	102976946
25	8424788	10841282	18657454	37433466	80331385	175877782
26	13913192	17909533	30901434	62337628	134873275	298279841
27	22847028	29416966	50884452	103186612	225015223	502496682
28	37315678	48055443	83327163	169824540	373141724	841161007
29	60631940	78093926	135733071	277967860	615224276	1399559416
30	98030644	126276743	219978688	452594316	1008792896	2315201903
31	157743554	203211038	354780782	733229626	1645443771	3808746574
32	252671288	325518314	569519349	1182159039	2670372299	6232651705
33	402944731	519138982	910130189	1897140990	4312780664	10147431024
34	639871871	824414851	1448166991	3031012912	6933014899	16440685315
35	1011956958	1303853212	2294680459	4821835750	11095408859	26512248644
36	1594100512	2053981256	3621419828	7639072393	17680429741	42561099330
37	2501559132	3223352798	5693103210	12054120068	28056800955	68028465562
38	3911136893	5039865872	8916408778	18947689292	44344779210	108279807765
39	6093172867	7852029282	13914109052	29672809254	69817667843	171651101620
40	9459795828	12191192807	21636960372	46301523560	109512215347	271048865628
41	14637397882	18865058704	33532084406	71997231090	171153951432	426389614752
42	22575337525	29097916032	51795716561	111575067538	266555833407	668307945618
43	34708392976	44740293582	79751566012	172343093538	413726582860	1043776858764
44	53199143209	68581738911	122415827920	265361653081	640040090348	1624602354318
45	81298470388	104816149708	187338790559	407324082526	986987600985	2520227376672
46	123880767618	159732599729				
47	188236334008	242738329372				
48	285242287944	367870426468				
49	431088527694	556024400588				
50	649816920320	838232884647				
51	977048352353					
52	1465442861255					
53	2192681711158					
54	3273114322046					
55	4874718706124					
56	7243754365560					
57	10740528588174					
58	15891194045343					
59	23462627747108					
60	34570490892429					

Table 3: Results from Exact Enumeration

## B A class of restricted plane partitions

A plane partition is an array of non-negative integers  $h_{i,j}$  that are weakly decreasing along both rows and columns i.e.,

$$h_{i+i,j} \leq h_{i,j} \quad \text{and} \quad h_{i,j+1} \leq h_{i,j} \quad \text{for all } i, j \geq 1 .$$

The volume of a plane partition is defined to be the sum of all entries in the array i.e.  $\sum_{i,j} h_{i,j}$  and let  $p_2(n)$  denote the number of plane partitions with volume  $n$ . Let  $pr(n)$  denote the subset of plane partitions where one imposes the stronger condition

$$h_{i+i,j} = h_{i,j} - e \quad \text{and} \quad h_{i,j+1} = h_{i,j} - e \quad \text{for all } i, j \geq 1 ,$$

where  $e = 0$  or  $e = 1$ .

The first few numbers are

$n$	1	2	3	4	5	6	7	8	9	10
$pr(n)$	1	2	3	6	10	18	30	41	63	102
$p_2(n)$	1	3	6	13	24	48	86	160	282	500

It is easy to see that

$$pr(n) < p_2(n) \text{ for } n > 1 . \quad (26)$$

We also have for  $n \gg 1$  that

$$\log a_1(n) < 4 \log pr(n/4).$$

We thus have

$$\log pr\left(\frac{n}{4}\right) > \frac{c_0}{4} n^{2/3} \quad \text{or} \quad \log pr(n) > \frac{c_0}{4^{1/3}} n^{2/3} . \quad (27)$$

We thus obtain the asymptotic bound as  $n \rightarrow \infty$ :

$$\frac{1}{4^{1/3}} n^{-2/3} \log a_1(n) < n^{-2/3} \log pr(n) < n^{-2/3} \log p_2(n) , \quad (28)$$

or equivalently

$$\boxed{\frac{c_0}{4^{1/3}} < n^{-2/3} \log pr(n) < \frac{3}{2}(2\zeta(3))^{1/3} .} \quad (29)$$

Our estimate of  $c_0 \approx 2.344$  thus enables us to set a lower bound for the asymptotic behaviour of  $pr(n)$ .

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